ON THE LOCATION OF THE ROOTS OF THE JACOBIAN OF TWO

BINARY FORMS, AND OF THE DERIVATIVE OF A

RATIONAL FUNCTION*

BY

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INTRODUCTION

Professor Bôcher has shown how the roots of certain algebraic invariants can be determined as the positions of equilibrium in the field of force due to properly situated repelling and attracting particles.† He considers a number of fixed particles either in a plane or on the surface of a sphere (the stereographic projection of the plane) and each of these particles is supposed to repel with a force equal to its mass divided by the distance. If a particle has negative mass, it attracts instead of repelling. The plane of the particles can be considered as the Gauss plane, and with this convention Bôcher proves the following theorem:‡

THEOREM I. *The vanishing of the jacobian of two binary forms f_1 and f_2 of degrees p_1 and p_2 respectively determines the points of equilibrium in the field of force due to p_1 particles of mass p_2 situated at the roots of f_1, and p_2 particles of mass $- p_1$ situated at the roots of f_2.*

Perhaps it is desirable briefly to indicate the proof of this theorem. We give the proof merely for the plane field. Let fixed particles of masses m_1, m_2, $\cdots$, m_n be placed at the points represented by the complex quantities $e_1, e_2, \cdots, e_n$ respectively. Then at any point x of the plane, the force due to these fixed particles is in magnitude, direction, and sense

$$K \left(\frac{m_1}{x - e_1} + \frac{m_2}{x - e_2} + \cdots + \frac{m_n}{x - e_n} \right),$$

where the symbol K indicates the conjugate of the complex quantity following.

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* Presented to the Society, February 23, 1918.

† Maxime Bôcher, *A problem in statics and its relation to certain algebraic invariants*, P r o c e e d i n g s   o f   t h e   A m e r i c a n   A c a d e m y   o f   A r t s   a n d   S c i e n c e s, vol. 40 (1904), p. 469. I am indebted to Professor Bôcher for a number of suggestions concerning the present paper.

‡ L. c., p. 476.

If homogeneous variables are introduced by the formulas

$$x = \frac{x_1}{x_2}, \qquad e_i = \frac{e'_i}{e''_i},$$

the plane field described in Theorem I becomes

$$K\left[x_2 \left(p_2 \sum_{i=1}^{i=p_1} \frac{e''_i}{e''_i x_1 - e'_i x_2} - p_1 \sum_{i=p_1+1}^{i=p_1+p_2} \frac{e''_i}{e''_i x_1 - e'_i x_2} \right) \right]$$

(1)

$$= K\left[x_2 \left[p_2 \frac{\frac{\partial f_1}{\partial x_1}}{f_1} - p_1 \frac{\frac{\partial f_2}{\partial x_1}}{f_2} \right] \right],$$

where

$$f_1 = (e''_1 x_1 - e'_1 x_2) \cdots (e''_{p_1} x_1 - e'_{p_1} x_2),$$

$$f_2 = (e''_{p_1+1} x_1 - e'_{p_1+1} x_2) \cdots (e''_{p_1+p_2} x_1 - e'_{p_1+p_2} x_2).$$

The quantity in the brackets in (1) reduces to the quotient of the jacobian of f_1 and f_2 by $f_1 f_2$, when Euler's theorem for homogeneous functions is applied. This completes the proof of Theorem I. It is to be noted that the jacobian vanishes not only at the points of no force, but also at the multiple roots of either form or a common root of the two forms; such a point is called a point of *pseudo-equilibrium*.

From the mechanical interpretation of Theorem I, Bôcher derives a number of results concerning the location of the roots of the jacobian with reference to the location of the roots of the ground forms.*

When we consider the mechanical system, it is intuitively obvious that there can be no position of equilibrium very near any of the fixed particles. In the Corollary to Theorem II of the present paper there is determined explicitly (and in an infinite variety of ways) a circle which can be drawn separating any one of these particles from the roots of the jacobian. If we have not one fixed particle but k particles, all attracting or all repelling, and if the remaining particles in the plane (or on the sphere) are sufficiently removed from those, then the mechanical system would lead us to expect that there could be no roots of the jacobian outside of and very near to a circle surrounding the k particles. This is a rough indication of the considerations that lead to Theorem II.

In the latter part of the paper some applications of these results are made to the roots of the derivative of a rational function.

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* See, e. g., Theorem III below.

PART I

Let us consider the statical system in the plane due to fixed particles of the kind described. We shall make use of two lemmas, which, indeed, are more general than is necessary for our use.

LEMMA I. *If Q is a point exterior to the circle C whose center is O, then of all possible positions for a unit (repelling or attracting) particle on or within C, that position nearest to Q causes the particle to exert the greatest force at Q,— greatest not only in magnitude but also in component along QO. That position farthest from Q causes the particle to exert the least force at Q,—least not only in magnitude but also in component along QO.*

LEMMA II. *If Q is a point interior to the circle C whose center is O, then of all possible positions for a unit (repelling or attracting) particle on or outside of C, that position nearest to Q causes the particle to exert the greatest force at Q,— greatest not only in magnitude but also in component along QO. Of all possible positions for an attracting particle on or outside of C, that position on C which is farthest from Q causes the particle to exert the force at Q which has the greatest component in the direction and sense QO.*

The truth of each of these lemmas becomes evident upon inverting C in the circle of unit radius and center Q, noting that the force exerted at Q by a unit particle at R is in direction and magnitude $R'Q$, where R' is the inverse of R in the unit circle whose center is Q.

We shall now apply these lemmas to Theorem I. Suppose there is in the plane a circle C_1 which contains on or within its circumference k roots of f_1. Suppose there is a circle C_2—larger than C_1 and concentric with it— outside of which lie all the remaining $p_1 - k$ roots of f_1. Suppose further that there is a circle C_3—also larger than C_1 and concentric with it—outside of which lie all the roots of f_2. Then we shall try to determine a circle C_0 larger than C_1 and concentric with it and such that there is no root of the jacobian of f_1 and f_2 within the annular region between C_0 and C_1.

We denote by O the common center of C_1, C_2, and C_3, and the radii of these circles by a, b, and c, respectively. We have supposed that $a < b$, $a < c$. Set up the statical system of Theorem I and consider the force at a point Q between C_1 and the smaller of C_2 and C_3. The component in the direction and sense OQ of the force due to the k positive particles (each of mass p_2) on or within C is not less than $p_2 k/(a + r)$, where r is the distance OQ. The component in the direction and sense QO of the force due to the positive particles outside of C_2 (whose mass is $(p_1 - k)p_2$) is not greater than $(p_1 - k)p_2/(b - r)$. The component in the direction and sense QO of the negative particles outside of C_3 (whose mass is $- p_1 p_2$) is not greater than

$p_1 p_2/(c + r)$.   If Q is a point of equilibrium, we must have

(2)
$$\frac{p_2 k}{a + r} \lessgtr \frac{(p_1 - k)p_2}{b - r} + \frac{p_1 p_2}{c + r},$$

(3)
$$\frac{kbc - p_1 ab - (p_1 - k)ac}{(p_1 - k)b + p_1 c - ka} \lessgtr r.$$

If the left-hand member of (3) is positive, we construct the circle with that radius and center O, and denote this circle by C_0. Then *it is readily seen that C_0 always lies within C_2 (unless $k = p_1$, when C_0 and C_2 coincide), and C_0 may or may not lie within C_1 and may or may not lie within C_3.* If C_0 lies outside of C_1 but within C_3, then we have shown that the annular region between C_0 and C_1 contains no point of equilibrium. This region contains no root of either form and therefore no possible point of pseudo-equilibrium. Hence the annular region contains no root of the jacobian of the forms. If on the other hand, C_0 lies outside of C_3, then between C_1 and C_3 there is no root of the jacobian of the forms.

If C_0 lies outside of C_1, it is readily shown that there are precisely $k - 1$ roots of the jacobian on or within C_1. Let the k roots of f_1 that are on or within C_1 move continuously so as to coincide at the point O, while the other roots of f_1 and all the roots of f_2 remain fixed. If Q is a position of equilibrium, inequality (2) obtains whenever Q is anywhere within C_1. Hence by (3) there is no root of the jacobian within C_1 except at O; and O is a $(k - 1)$-fold root. During the change of the k roots of f_1 the roots of the jacobian change continuously (at least when we refer to the sphere instead of the plane) and are never in the annular region between C_1 and the nearer of C_0 and C_3. Hence at the start there were just $k - 1$ roots of the jacobian on or within C_1.

Let us determine the circle C_0 by invariant elements. Suppose a line through O cuts the circles C_i in the points C_i' and C_i'' $(i = 0, 1, 2, 3)$ where the notation is such that O separates no pair of points C_i', C_j'. We find that*

$$(C_1'', C_2', C_3'', C_0') = p_1/k.$$

Hence, for the special case that C_1, C_2, and C_3 are concentric, with C_1 in the interior of C_2 and C_3, we have proved:

THEOREM II. *Suppose that f_1 and f_2 are two binary forms, the degree of f_1 being p_1, and suppose there are k roots of f_1 which lie in a closed region T_1 bounded by a circle C_1. Suppose there is a second closed region T_2 bounded by a circle C_2,*

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* We are using the following definition for the cross-ratio:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

that T_2 has no point in common with T_1, and that T_2 contains the remaining $p_1 - k$ roots of f_1. Suppose further that there is a third closed region T_3 bounded by a circle C_3 coaxial with C_1 and C_2, that T_3 has no point in common with T_1, and that T_3 contains all the roots of f_2.

1. *If the circle C_0 described below lies in the region between C_1 and C_3, then there are no roots of the jacobian of f_1 and f_2 in the region included between C_1 and C_0; furthermore, there are just $k - 1$ roots of the jacobian in T_1.*

2. *If the circle C_0 lies in the region between C_2 and C_3, then there are no roots of the jacobian of f_1 and f_2 in the region included between C_1 and C_3; moreover, there are just $k - 1$ roots of the jacobian in T_1.*

In this theorem, C_0 denotes that circle of the coaxial family to which C_1, C_2, and C_3 belong which is the locus of points C_0' such that

$$(C_1'', C_2', C_3'', C_0') = p_1/k.$$

C_i' and C_j'' denote the points in which any circle T orthogonal to the circles of the family cuts the circle C_i, and the notation is such that on T, neither null circle of the family shall separate any of the pairs of points C_i', C_j' $(i, j = 0, 1, 2, 3)$. *

This theorem is proved for the case that C_1, C_2, and C_3 are not concentric by making a linear transformation that transforms them into concentric circles, with C_1 in the interior of C_2 and C_3. (Such a transformation always exists.) Since the theorem is true for this particular case, and since everything used in the theorem is invariant under linear transformation, the theorem is true as stated.†

It is also true that Theorem II refers to the sphere as well as the plane, for everything essential in the theorem is invariant under stereographic projection.

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* Reference to the italicized sentence immediately below (3) will show that 1 and 2 cannot occur at the same time. It may occur that C_0 lies in neither of these positions, which in the case of concentric circles means that C_0 lies within C_1; if this is true, the theorem makes no statement about the roots of the jacobian. Also, the circle C_0 may not exist, which means that the left-hand side of (3) is negative.

If no root of f_1 lies on C_1, if no root of f_1 lies on C_2 ($k \neq p_1$), or if no root of f_2 lies on C_3, then in case (1) no root of the jacobian can lie on C_0. This is immediately seen by omitting the equality sign in (2) and hence in (3).

Of course, a theorem similar to II can be proved for Bôcher's covariant ϕ (l. c., p. 474).

Theorem II can be applied to the roots of special types of polynomials, but as Professor Curtiss pointed out to me, the following more general theorem can be proved by means of Lemma II. This more general theorem is a special case of the theorem just suggested concerning the covariant φ.

If $f(z)$ is a polynomial of degree n all of whose roots lie outside of a circle whose center is the origin and radius b, and if k_1 and k_2 are any positive numbers, then all the roots of $k_1 z f'(z) - k_2 f(z)$ lie outside the smaller of the two circles whose common center is the origin and whose radii are b and $k_2 b / (nk_1 - k_2)$ respectively. See Laguerre, Œuvres, vol. I, pp. 56, 133; see also the reference to Gonggrÿp below.

† We consider the exterior of a circle, including the boundary and the point at infinity, to be a closed region.

It is readily shown that Theorem II gives in general the largest region which will be free from roots of the jacobians of all pairs of forms which satisfy the hypothesis. Let us take the circles C_1, C_2, and C_3 in their original (concentric) positions, and first suppose C_0 to lie between C_1 and C_3. Then reference to inequality (2) shows that if $k \neq p_1$, the position of the particles which determine the field of force can be chosen so that (2) becomes an *equality*, and there will be a position of equilibrium on C_0. If $k = p$, C_0 and C_2 coincide, and we can consider C_2 to coincide with C_3. In this case, or if on the other hand C_0 lies outside of C_3, there can be chosen on C_3 a multiple root of f_2, which will be a root of the jacobian.

If we take C_1 a null circle P, and if we let C_2 and C_3 coincide and denote this circle by C, we have the following result:

COROLLARY. *Suppose that f_1 and f_2 are two binary forms, the degree of f_1 being p_1, and suppose that the circle C separates P (a k-fold root of f_1) from those roots of f_2 and f_1 (other than P) which do not lie on C itself. Then the circle C_0 separates P from those roots of the jacobian of f_1 and f_2 (other than P) which do not lie on C_0 itself, where C_0 is that circle of the coaxial family determined by C and P which is the locus of points C_0' such that*

$$(P, C', C'', C_0') = p_1/k;$$

C' and C'' denote the intersections of C with the circle through P and C_0' orthogonal to C.

In the corollary, it is of course true that C_0 lies between P and C unless $k = p_1$, when C_0 and C coincide.

If we take $k = p_1$, and if C_2 and C_3 are chosen coincident, Theorem II gives the following theorem, which is due to Bôcher:[*]

THEOREM III. *If the roots of a binary form f_1 of degree p_1 lie in a closed region T_1 and if the roots of a second binary form f_2 of degree p_2 lie in a second closed region T_2 which has no point in common with T_1, and if these two regions are bounded by arcs of circles each one of which circles separates the interior of T_1 from the interior of T_2, then the jacobian of f_1 and f_2 has just $p_1 - 1$ roots in T_1 and $p_2 - 1$ roots in T_2.*[†]

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[*] As Professor Curtiss pointed out to me, the statement given by Bôcher (l. c., p. 478) is not quite accurate. This inaccuracy has been here corrected.

[†] There have recently been published two results which are special cases of Theorem III, although the authors were apparently not aware of the fact.

See L. R. Ford, *On the roots of a derivative of a rational function*, Proceedings of the Edinburgh Mathematical Society, vol. 33 (1915). Several of Ford's results are generalized in the present paper.

See also B. Gonggrÿp, *Quelques théorèmes*, etc., Liouville's Journal, ser. 7, vol. 1 (1915), p. 360. Compare the former reference to Laguerre.

PART II

All the theorems concerning jacobians which were proved by Bôcher, as well as the theorems of the present paper can be immediately applied to the roots of the derivative of a rational function.

Let us take any rational function not a constant, $f(z) = u(z)/v(z)$, and suppose (as we can do with no loss of generality) that u and v have no common factor containing z. Introduce homogeneous coördinates, setting $z = z_1/z_2$, and multiply the numerator and denominator of f by z_2^n, where n is the degree of f:[*]

$$f(z) = \frac{z_2^n u(z_1/z_2)}{z_2^n v(z_1/z_2)} = \frac{f_1(z_1, z_2)}{f_2(z_1, z_2)}.$$

If we express $f'(z)$, the derivative of $f(z)$, in terms of J, the jacobian of f_1 and f_2, we find

$$f'(z) = \frac{J}{n}\left(\frac{z_2}{f_2}\right)^2.$$

From this relation it follows that *the roots of f' are the roots of J and a double root at infinity, except that when one of these points is also a pole of f it cannot be a root of f'.*

We shall not attempt to carry over all the results concerning the roots of the jacobian to the corresponding results for the derivative of a rational function. We merely give a few examples by way of illustration.[†] The following theorem is a direct application of Theorem III.

If $f(z)$ is a rational function of degree n whose roots lie in a closed region T_1 and whose poles lie in a second closed region T_2 which has no point in common with T_1, and if these two regions are bounded by arcs of circles each one of which circles separates the interior of T_1 from the interior of T_2; then all the roots of the derivative of $f(z)$ lie in T_1 and T_2, except that there are two additional roots at infinity if $f(z)$ has no pole there. Except for these two possible roots, there are just $n - 1$ roots of $f'(z)$ in T_1, and if $f(z)$ has no multiple pole there are just $n - 1$ roots of $f'(z)$ in T_2.[‡]

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[*] The degree of a rational function is the greater of the degrees of its numerator and denominator, or the common degree if the numerator and denominator have the same degree.

[†] Essentially the following theorem is given by Bôcher (l. c., p. 479): " If f_1 and f_2 are two forms and if all the roots of each form either lie on a circle C or are situated in pairs of points inverse with respect to C, then all the roots of the jacobian of f_1 and f_2 also lie on C or are situated in pairs of points inverse with respect to C. On any arc of C bounded by roots of f_1 (or of f_2) and containing no root of either form there is at least one root of the jacobian."

It is true that the force at any point of C (when the statical system is set up) is in direction tangent to C. Hence, if there are *two* circles C of the theorem stated in this footnote, their intersection must be a root of the jacobian or a root of one of the ground forms.

Both of these theorems can evidently be extended to the derivative of a rational function.

[‡] This is a generalization of the well-known theorem of Lucas that " the roots of the deriva-

We can immediately obtain an upper bound for the moduli of the finite roots of the derivative of a rational function. Suppose f to have m_1 finite roots (or poles) and m_2 finite poles (or roots), $m_1 > m_2$. It follows from the corollary to Theorem II that if a circle whose radius is a includes all the finite roots and poles of f, then a concentric circle of radius $a\,(m_1 + m_2)/(m_1 - m_2)$ includes all the finite roots of f'.

CAMBRIDGE, MASS.
January, 1918

tive of a polynomial lie within or on the boundary of the smallest convex polygon enclosing the roots of the original polynomial."

As Bôcher points out, Theorem III is also a generalization of Lucas's theorem.

The following theorem is a corollary of Theorem II: *If $f(z)$ is a polynomial of degree n which has a k-fold root at P, and if a circle whose center is P and radius a includes no root of f other than P, then the circle whose center is P and radius $ak \mid n$ includes no root of $f'(z)$ other than P.*